On the Problem of Programming Quantum Computers

Hans De Raedt, Anthony Hams, and Kristel Michielsen
Institute for Theoretical Physics and Materials Science Centre
University of Groningen, Nijenborgh 4
NL-9747 AG Groningen, The Netherlands
E-mail: deraedt@phys.rug.nl, A.H.Hams@phys.rug.nl, kristel@phys.rug.nl
http://rugth30.phys.rug.nl/compphys

Seiji Miyashita and Keiji Saito
Department of Applied Physics, School of Engineering
University of Tokyo, Bunkyo-ku, Tokyo 113, Japan
E-mail: miya@yuragi.t.u-tokyo.ac.jp, saitoh@spin.t.u-tokyo.ac.jp
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Abstract

We study effects of the physical realization of quantum computers on their logical operation. Through simulation of physical models of quantum computer hardware, we analyse the difficulties that are encountered in programming physical implementations of quantum computers. We discuss the origin of the instabilities of quantum algorithms and explore physical mechanisms to enlarge the region(s) of stable operation.

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I. INTRODUCTION

The logical operation of a conventional digital computer does not depend on the details of the hardware implementation although the speed of operation and the cost of the machine obviously do. Conventional computers are in one particular state at any time. Furthermore from the point of view of programming the computer, the internal machinery of the basic units comprising the computer is irrelevant. This is very important because it implies that on a conceptual level, algorithms designed to run on a conventional computer will produce answers that do not depend on the hardware that is used.

A quantum computer (QC) is very different in this respect. A QC exploits the fact that a quantum system can be in a superposition of states and that interference of these states allows exponentially many computations to be done in parallel [?,?,?,?,?]. The presence of this superposition is a manifestation of the internal quantum dynamics of the elementary units (i.e. the qubits). In other words, the quantum dynamics is essential to the operation of a physically realizable QC.

The operation of an ideal QC does not depend on the intrinsic dynamics of the physical qubits: One imagines that the qubits are ideal two-state quantum systems that perform their task instantaneously and perfectly. From a theoretical point of view this situation is very similar to that of computers built from conventional digital circuits. However, in practice there is a fundamental difference. The fact that the logical operation of conventional digital circuits does not depend on their hardware implementation (e.g. semiconductors, relays, vacuum tubes, etc.) is directly related to the presence of dissipative processes that drive the circuits into regions of stable operation. Dissipation suppresses the effects of the internal, non-ideal (chaotic) dynamics of these circuits.

The quantum dynamics of small physical devices is usually very sensitive to small perturbations and this holds for qubits as well. Unfortunately, in contrast to the case of conventional circuits, dissipation usually has a devastating effect on the coherent quantum dynamical motion of the qubits, i.e. on the very essence of QC's. Therefore the specific physical realization of a QC is intimately related to the stability of its operation.

In this paper we study the relation between the physical realization of QC's and their logical operation and explore physical mechanisms to enlarge the region(s) of stable operation. We demonstrate that programming a physical implementation of a QC is non-trivial, even if the QC consists of only two or four qubits. In most theoretical work on QC's and quantum algorithms (QA's) [?,?,?,?,?,?,?,?,?,?] one considers theoretically ideal (but physically unrealizable) QC's and therefore this problem of programming QC's (which we will call the Quantum Programming Problem (QPP)) is not an issue. As far as we know no experimental data has been published that specifically addresses this, for potential applications, very important and intrinsic problem of programming QC's. The aim of this paper is to investigate various aspects of the QPP by simulating QC hardware.

How does a QPP reveal itself? Consider two logically independent operations (O_1 and O_2) of the machine. On a conventional computer or ideal QC, the order in which we execute these two mutually independent instructions does not matter: $O_1O_2 = O_2O_1$. However, it turns out that on a physically realizable QC *sometimes* the order does matter, even if there are no dependencies in these two program steps. In some cases $O_1O_2 \neq O_2O_1$ and the QC may give the wrong answers. The QPP is due to the fact that we are dealing with

interacting quantum mechanical objects (as communication between qubits is essential for computation), technical difficulties to manipulate a qubit without disturbing others and the fundamental physical fact that the state of a qubit cannot be frozen during the time that other qubits are being addressed. Also note the qualifier *sometimes*. There seems to be no general rule to decide beforehand which operation and at what stage of the QA the QPP leads to incorrect results. At present the only way to find out seems to be to actually carry out the calculations and check the results.

It does not require a lot of imagination to see that the QPP implies that it may be very difficult to develop a non-trivial quantum program for a physical QC. Moreover there is no guarantee that a QA that works well on one QC will perform well on other QC's. Marginal changes in the qubit hardware may affect the interchangeability of logically independent operations. There are several factors that contribute to the QPP:

1) Differences between the theoretically perfect and physically realizable one- and two-qubit operations, e.g. the one-qubit operations affecting other qubits and inaccuracies on the time-interval used to perform operations.

2) Physical qubits cannot be kept still while others are being addressed.

3) The effect of coupling of the qubits to other degrees of freedom (dissipation, decoherence).

In this paper we address these issues through case studies. In Section II we describe the physical model that will be the starting point of our investigations. Our choice of physical models is largely inspired by NMR-QC experiments [**?**,**?**,**?**,**?**,**?**,**?**], only because other candidate technologies [**?**,**?**,**?**,**?**,**?**,**?**,**?**,**?**,**?**,**?**,**?**,**?**,**?**,**?**,**?**] for building QC's are not yet developed to the point that they can execute computationally non-trivial QA's.

As an example of such a QA we will take Grover's database search algorithm (see Section III) and implement it on various physical models for 2- and 4-qubit QC's (Sections IV to **??**). Our approach for analysing the QPP is to run Grover's QA by simulating the time-evolution of the physical model representing the QC. Thereby we strictly stick to the rules of quantum mechanics, i.e. we solve the time-dependent Schrödinger equation that describes the evolution of the physical apparatus representing the QC. The main vehicle for performing these simulations is a Quantum Computer Emulator (QCE) [**?**]. A detailed description of this software tool is given elsewhere [**?**,**?**]. Our work is fundamentally different from those of others who also address questions related to error propagation in QA's [**?**,**?**,**?**,**?**] in that we execute QA's on physical models of QC hardware. For simplicity most of our calculations (Sections IV and **??**) are done for systems at zero temperature, in the absence of interactions with the environment. Simulations of QC's operating at a non-zero temperature, in contact with a heat bath, are discussed in Section **??**.

II. PHYSICAL MODEL

The simplest qubit is a two-state quantum system, e.g. the spin of electrons or the polarization of photons. The basic operations in a meaningful computation are the manipulation of each qubit (e.g. by applying external fields) and the exchange of information between the

qubits. In physical terms, the latter implies that there should be some interaction between the qubits. A non-trivial QC contains at least two qubits. It is known that the most simple spin-1/2 system, i.e. the Ising model, can be used for quantum computing [**?**,**?**,**?**,**?**]. In the presence of an external magnetic field, the Hamiltonian of the two-spin Ising model reads

$$H = -J_z S_1^z S_2^z - h_z(g_1 S_1^z + g_2 S_2^z), \tag{1}$$

where J_z, g_1 and g_2 are material-specific constants, h_z represents the applied magnetic field, and S_j^α denotes the α-th component ($\alpha = x, y, z$) of the spin-1/2 operators describing the nuclear spins. In this paper we use units such that $\hbar = 1$.

According to the rules of quantum mechanics, the state of the QC at time t is completely described by the wave function $|\Phi(t)\rangle$. Executing a program on a QC is equivalent to solving the time-dependent Schrödinger equation (TDSE)

$$i\frac{\partial}{\partial t}|\Phi(t)\rangle = H(t)|\Phi(t)\rangle. \tag{2}$$

For the 2-qubit QC the most general linear combination reads

$$|\Phi(t)\rangle = a(\downarrow,\downarrow;t)|\downarrow\downarrow\rangle + a(\uparrow,\downarrow;t)|\uparrow\downarrow\rangle + a(\downarrow,\uparrow;t)|\downarrow\uparrow\rangle + a(\uparrow,\uparrow;t)|\uparrow\uparrow\rangle, \tag{3}$$

where the $a(b_1, b_2; t)$ are complex numbers. The probability that, upon measurement, the QC is in one of the four basis states $|\downarrow\downarrow\rangle, \ldots, |\uparrow\uparrow\rangle$ is given by $|a(\downarrow,\downarrow;t)|^2, \ldots, |a(\uparrow,\uparrow;t)|^2$ respectively.

The final results of a QC calculation can be read off by performing an experiment that measures the expectation value(s) of the spin(s). The value of a qubit is related to the expectation value of the z-component of the spin operator:

$$Q_j \equiv 1/2 - \langle\Phi(t)|S_j^z|\Phi(t)\rangle \quad ; \quad j = 1, 2, \ldots, N, \tag{4}$$

where N denotes the total number of qubits of the QC. In this paper we will denote the state of a qubit by

$$|0\rangle \equiv |\uparrow\rangle \quad ; \quad |1\rangle \equiv |\downarrow\rangle, \tag{5}$$

and if necessary we will add a subscript, e.g. $|0\rangle_1$, to label the qubit.

The Ising interaction between the spins is sufficient to implement control-NOT (CNOT) gates. Single qubit operations can be performed by applying additional external fields in the x and y direction (see below). It has been shown that any operation involving an arbitrary number of qubits can be written as a sequence of these elementary operations [**?**], in other words the Ising model is a universal QC.

III. GROVER'S DATABASE SEARCH ALGORITHM

Finding the needle in a haystack of N elements requires $\mathcal{O}(N)$ queries on a conventional computer [**?**]. Grover has shown that a QC can find the needle using only $\mathcal{O}(\sqrt{N})$ attempts [**?**,**?**]. Assuming a uniform probability distribution for the needle, for $N = 4$ the average number of queries required by a conventional algorithm is 9/4 [**?**,**?**]. With Grover's QA the

correct answer can be found in a single query [**?**,**?**]. This QA has been implemented on a 2-qubit NMR-QC for the case of a database containing four items [**?**,**?**]. This implementation uses elementary rotations about 90 degrees (clock-wise) around the x and y-axis (e.g. $X_1|00\rangle = (|00\rangle + i|10\rangle)/\sqrt{2}$ etc.) and an interaction-controlled phase shift [**?**,**?**]. In matrix notation we have, for example,

$$X_1 \begin{pmatrix} |00\rangle \\ |10\rangle \\ |01\rangle \\ |11\rangle \end{pmatrix} \equiv \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i & 0 & 0 \\ i & 1 & 0 & 0 \\ 0 & 0 & 1 & i \\ 0 & 0 & i & 1 \end{pmatrix} \begin{pmatrix} |00\rangle \\ |10\rangle \\ |01\rangle \\ |11\rangle \end{pmatrix}, \tag{6}$$

where $|b_1 b_2\rangle \equiv |b_1\rangle|b_2\rangle$ and $b_i = 0, 1$.

$$Y_2 \begin{pmatrix} |00\rangle \\ |10\rangle \\ |01\rangle \\ |11\rangle \end{pmatrix} \equiv \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{pmatrix} \begin{pmatrix} |00\rangle \\ |10\rangle \\ |01\rangle \\ |11\rangle \end{pmatrix}. \tag{7}$$

The two qubits "communicate" with each other through the interaction-controlled phase shift

$$I(a) \begin{pmatrix} |00\rangle \\ |10\rangle \\ |01\rangle \\ |11\rangle \end{pmatrix} \equiv \frac{1}{\sqrt{2}} \begin{pmatrix} e^{-ia/4} & 0 & 0 & 0 \\ 0 & e^{+ia/4} & 0 & 0 \\ 0 & 0 & e^{+ia/4} & 0 \\ 0 & 0 & 0 & e^{-ia/4} \end{pmatrix} \begin{pmatrix} |00\rangle \\ |10\rangle \\ |01\rangle \\ |11\rangle \end{pmatrix}. \tag{8}$$

This unitary transformation can be used to implement a CNOT gate and hence (6),(7), and (8) can be used to construct a universal QC.

To see how Grover's QA works it is instructive to consider an example of a database with $N = 4$ positions, labeled 0,...,3. Let us assume that the item we are searching for is located at position 2. First we put the QC in its initial state $|00\rangle$. Then we transform this state into the uniform superposition state

$$|u\rangle \equiv \frac{1}{2}(|00\rangle + |10\rangle + |01\rangle + |11\rangle), \tag{9}$$

by letting the sequence $\overline{X}_2\overline{X}_2\overline{Y}_2\overline{X}_1\overline{X}_1\overline{Y}_1$ act on $|00\rangle$ [**?**]:

$$|u\rangle = \overline{X}_2\overline{X}_2\overline{Y}_2\overline{X}_1\overline{X}_1\overline{Y}_1|00\rangle, \tag{10}$$

where $\overline{X}_j$ ($\overline{Y}_j$) denotes the inverse of X_j (Y_j).

Next we apply to $|u\rangle$ the sequence of elementary operations [**?**,**?**,**?**,**?**]

$$F_2 \equiv \overline{Y}_1 X_1 \overline{Y}_1 Y_2 \overline{X}_2 \overline{Y}_2 I(\pi), \tag{11}$$

to encode the content of the database as

$$|\Psi_2\rangle = \frac{1}{2}(|00\rangle + |10\rangle - |01\rangle + |11\rangle), \tag{12}$$

where we adopt the notation in which the basis states are labeled by the binary representation of integers with the order of the bits reversed. In (12) the position of the item in the database (i.e. 2 in this example) is encoded by modifying $|u\rangle$ such that the amplitude of the corresponding basis state changes sign.

The key ingredient of Grover's algorithm is an operation that determines which of the basis state contributes to (12) with the minus-one amplitude. In matrix notation

$$G = \frac{1}{2}\begin{pmatrix} -1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 \\ 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & -1 \end{pmatrix}, \tag{13}$$

and in terms of elementary operations

$$G = X_1 X_1 \overline{Y}_1 X_2 X_2 \overline{Y}_2 Y_1 \overline{X}_1 \overline{Y}_1 Y_2 \overline{X}_2 \overline{Y}_2 I(\pi) X_1 X_1 \overline{Y}_1 X_2 X_2 \overline{Y}_2. \tag{14}$$

The sequence (14) is by no means unique: Various alternative expressions can be written down by using the algebraic properties of the X's and Y's. This feature can be exploited to eliminate redundant elementary operations [**?**]. Starting from the uniform superposition state $|u\rangle$, one choice for the optimized sequences that implement the four different states of the database and Grover's search algorithm is [**?**,**?**,**?**,**?**] .ΩΩSee the amstex package documentation for explanation.ΩType~H¡return¿~for~immediate~helpeq:sequence

$$U_0 = X_1 \overline{Y}_1 X_2 \overline{Y}_2 I(\pi) X_1 \overline{Y}_1 X_2 \overline{Y}_2 I(\pi), \tag{15a}$$
$$U_1 = X_1 \overline{Y}_1 X_2 \overline{Y}_2 I(\pi) X_1 \overline{Y}_1 \overline{X}_2 \overline{Y}_2 I(\pi), \tag{15b}$$
$$U_2 = X_1 \overline{Y}_1 X_2 \overline{Y}_2 I(\pi) \overline{X}_1 \overline{Y}_1 X_2 \overline{Y}_2 I(\pi), \tag{15c}$$
$$U_3 = X_1 \overline{Y}_1 X_2 \overline{Y}_2 I(\pi) \overline{X}_1 \overline{Y}_1 \overline{X}_2 \overline{Y}_2 I(\pi), \tag{15d}$$

where the U_n correspond to the case where the needle is in position n.

On an ideal QC sequences (**??**) return the correct answer, i.e. the position of the searched-for item. This is easily verified on our Quantum Computer Emulator [**?**] by selecting the elementary operations (called micro instructions on the QCE) that implement an ideal QC.

IV. TWO-QUBIT QC'S

The energy-level structure of the nuclear spins of molecules such as deuterated cytosin [**?**,**?**,**?**] and carbon-13 labeled chloroform [**?**,**?**] can be described by model (1) and hence they can be used as physical realizations of 2-qubit QC's.

A. Resonant pulses

NMR-QC experiments on carbon-13 labeled chloroform [**?**] use resonant pulses to manipulate the quantum state of the nuclear spins of the ^{1}H and ^{13}C atoms. In the presence of a static magnetic field along the $+z$ direction this NMR-QC system is described by (1) with $h_z g_1/2\pi \approx 500$MHz, $h_z g_2/2\pi \approx 125$MHz, and $J_z/2\pi \approx -215$Hz [**?**]. A detailed account

of simulations for this case have been published elsewhere [**?**,**?**]. Simulations for physical model (1) confirm that sequences (**??**) yield the correct answers for the database search problem [**?**]. However, we also demonstrated that the outcome of these calculations may be very sensitive to the order in which logically independent operations are carried out [**?**].. Some of these results are reproduced in Table **??** (first six rows, for details see [**?**,**?**]).

The results marked with a tilde are obtained by using a logically identical but physically different uniform superposition, i.e.

$$|u'\rangle = \overline{X}_1\overline{X}_1\overline{Y}_1\overline{X}_2\overline{X}_2\overline{Y}_2|00\rangle. \tag{16}$$

On an ideal QC, $|u'\rangle = |u\rangle$ but on a physical realizable machine this is unlikely to be the case. In an experiment it is simply impossible to freeze spin 2 (1) during the time that resonant pulses are being applied to spin 1 (2). Unless the length of these pulses is chosen judisciously, the wave function will acquire an additional phase. The corresponding unitary transformation does not necessarily commute with the operations that follow, potentially leading to an incorrect final result (as shown in Table **??**), as we pointed out in a previous paper [**?**].

For the two-spin system (16) one may optimize the pulse durations such that the effect of these phase errors yields qualitatively correct answers. A basic step is to make the pulse lengths commensurate with all relevant time scales [**?**,**?**]. The results of the calculations are shown in Fig. **??** and summarized in Table **??** (primed symbols) [**?**]. It is clear that optimization has the desired effect on the sequences that operate on $|u'\rangle$ (symbols with a tilde).

In spite of this optimization, Table **??** shows that there are significant quantitative differences between the theoretically exact results (rows (1,2)) and those obtained by simulating a physical model of a QC (e.g. rows 7 to 10). Even if the pulse length is taken to be commensurate with the relevant time scales of the QC, changing the state of the qubits by way of resonant pulses yields quantum states that are different from those obtained by means of the unitary transformations used in the analysis of the ideal QC. This is because spin 2 also interacts with the field applied to spin 1 and vice versa. With each program step, the non-ideal unitary operation may or may not result in the prolifiration of errors. These errors are systematic (there is no "random" error source in our calculations) and directly linked to the structure of the QA. This is a clear case of a QPP, although we managed to let the different QA's produce the correct answer. Note that the QPP cannot be solved by means of error correction [**?**]: The operations on the extra qubits required for error correction will suffer from exactly the same QPP.

B. Stability of Grover's quantum algorithm

So far we studied the stability of quantum algorithms by perturbing the input to the database encoding part of the algorithm. In this subsection we will study the QPP of the database query part of Grover's search algorithm (the operation G, see (14)) itself. First we will asssume that the input provided to G is exact (i.e. of the form (10) for example) and we will compare the output of logically identical but physically different implementations of G. As examples we take the original sequence

$$G = X_1X_1\overline{Y}_1X_2X_2\overline{Y}_2Y_1\overline{X}_1\overline{Y}_1Y_2\overline{X}_2\overline{Y}_2I(\pi)X_1X_1\overline{Y}_1X_2X_2\overline{Y}_2, \tag{17}$$

and two, logically identical, sequences

$$\hat{G} = X_1 X_1 \overline{Y}_1 X_2 X_2 \overline{Y}_2 Y_1 \overline{X}_1 Y_2 \overline{X}_2 \overline{Y}_1 \overline{Y}_2 I(\pi) X_1 X_1 \overline{Y}_1 X_2 X_2 \overline{Y}_2, \tag{18}$$

$$\tilde{G} = X_1 X_1 \overline{Y}_1 X_2 X_2 \overline{Y}_2 Y_2 \overline{X}_2 \overline{Y}_2 Y_1 \overline{X}_1 \overline{Y}_1 I(\pi) X_2 X_2 \overline{Y}_2 X_1 X_1 \overline{Y}_1. \tag{19}$$

Note that on purpose we did not "optimize" these sequences by using e.g. $\overline{X}_1 X_1 = 1$. On an ideal QC we have [**?**]

$$G|\Psi_2\rangle = G^3|\Psi_2\rangle = \hat{G}|\Psi_2\rangle = \hat{G}^3|\Psi_2\rangle = \tilde{G}|\Psi_2\rangle = \tilde{G}^3|\Psi_2\rangle = |01\rangle, \tag{20}$$

providing another test of the stability of the query operation G on a physical QC. Table **??** contains the numerical results obtained by running the sequences (17), (18), and (19) on the QCE using the exact state $|\Psi_2\rangle$ (see (14)) as input. In the case of the NMR-like QC optimized resonant pulses were used. The ideal QC performs as expected but the physical implementation (symbols with a prime) does not. In fact even one application of $\tilde{G}$ apparently returns an answer that is close to being useless ($Q_1' \approx 0.5$). As the three sequences (17), (18), and (19) are logically identical this is a clear case of a QPP.

The occurence of a QPP seems to be a generic feature of QA's running on QC's. Therefore it is of interest to try to quantify the QPP. We now describe a simple procedure for this purpose, using G and the case where the item is located in position 0 (the exact input state being $|\Psi_0\rangle$) as an example. In general there are two sources of errors in this calculation: The input $|\Psi'\rangle$ to G and G itself, the latter depending on the particular hardware implementation of the QC. As before we will use the resonant pulse technique in our numerical experiments.

We write $|\Psi'\rangle$ as

$$|\Psi'\rangle = \alpha_0|\Psi_0\rangle + \alpha_1|\Psi_1\rangle + \alpha_2|\Psi_2\rangle + \alpha_3|\Psi_3\rangle, \tag{21}$$

where the amplitude α_0 can always be taken real ($-1 \leq \alpha_0 \leq 1$) and is chosen at random. The other three complex coeffients are chosen randomly too, subject to the constraint $|\alpha_1|^2 + |\alpha_2|^2 + |\alpha_3|^2 = 1 - |\alpha_0|^2$. The real variable $x \equiv \langle\Psi'|\Psi_0\rangle = \alpha_0$ is a measure for how much the input state deviates from the exact reference input $|\Psi_0\rangle$. On an ideal QC, $G|\Psi_0\rangle = |00\rangle$. Thus we can use the state $|00\rangle$ as reference to determine how much the output state $|\Theta\rangle \equiv G|\Psi'\rangle$ deviates from the exact answer. We quantify this deviation by the variable $y = |\langle 00|G\Psi'\rangle|$.

The result of a numerical experiment using 20000 random input states $|\Psi_0\rangle$ is shown in Fig. **??**. Plots for the three other cases are nearly identical. We classify input-output pairs as "good" or "bad" as follows. First we choose a confidence level $0 \leq c \leq 1$ ($c = 0.7$ for the data shown in Fig. **??**). A particular input-output pair is considered to be good if $x^2 \geq c$ and $y^2 \geq c$. In Fig. **??** the good (bad) pairs are shown by black (gray) markers.

At a fairly low confidence level of $c = 0.7$, the region of stable operation of the G operation is rather small. This corroborates our earlier finding that successive applications of G, e.g. G^3, rapidly drive the system into a region of instability. In general, quantum systems are very sensitive to noise and become more sensitive as the number of operations on qubits is increased [**?**]. In the absence of dissipation, it is easy for the system to leave the relatively small manifold of good input states, a charactistic feature of almost chaotic dynamics.

C. Hard nonselective pulses

Another physical implementation of a 2-qubit QC employs the nuclear spins of two ^{1}H spin-1/2 nuclei in a solution of cytosine in D_2O [?,?,?]. This system can also be described by Hamiltonian (1). In the NMR experiments hard nonselective pulses are used to address the qubits. In this section we study the stability of QC operation for this physical realization of a QC.

As usual it is expedient to transform to another frame of reference that rotates with a constant frequency. This is accomplished by substituting in the TDSE

$$|\Phi(t)\rangle = e^{ith_z(g_1+g_2)(S_1^z+S_2^z)/2}|\Psi(t)\rangle. \tag{22}$$

The time evolution of $|\Psi(t)\rangle$ is then governed by the Hamiltonian

$$H = -J_z S_1^z S_2^z - \frac{\Omega}{2}S_1^z + \frac{\Omega}{2}S_2^z, \tag{23}$$

where $\Omega = h_z(g_1 - g_2)$. Guided by experiment [?,?] in our numerical work we will set $\Omega/2 = 1$ and $J_z/\pi\Omega = -0.01887$ (in dimensionless units).

We now consider the time evolution of the two spins when we apply a static magnetic field h_x along the x-axis. The Hamiltonian in the laboratory frame is given by

$$H = -J_z S_1^z S_2^z - h_z(g_1 S_1^z + g_2 S_2^z) - h_x(g_1 S_1^x + g_2 S_2^x), \tag{24}$$

and the corresponding expression in the rotating frame of reference reads

$$\begin{aligned} H = &- J_z S_1^z S_2^z - \frac{\Omega}{2}S_1^z + \frac{\Omega}{2}S_2^z \\ &- h_x(g_1 S_1^x + g_2 S_2^x)\cos\Omega t - h_x(g_1 S_1^y + g_2 S_2^y)\sin\Omega t. \end{aligned} \tag{25}$$

If the duration of the pulse is short, i.e. $\Omega t \ll 1$, it is a good approximation to drop the time-dependence in (25) and we obtain

$$\widetilde{H} = -\frac{\Omega}{2}S_1^z + \frac{\Omega}{2}S_2^z - h_x(g_1 S_1^x + g_2 S_2^x), \tag{26}$$

where we used the fact that since $|J_z| \ll |\Omega/2|$, for short pulses the effect of the spin-spin interaction on the time evolution is small and may be neglected. It is instructive to compute the time evolution of the spins, initially in state $|0\rangle$ ($|0\rangle = |\uparrow\rangle$ by convention), under these circumstances. A straightforward calculation yields

$$Q_j = \langle 0|e^{it\widetilde{H}} S_j^z e^{-it\widetilde{H}}|0\rangle = \frac{1}{2} - \frac{[2h_x g_j(\Omega/2+\lambda)]^2}{[(\Omega/2+\lambda_j)^2 + h_x^2 g_j^2]^2}\sin^2\frac{\lambda_j t}{2}, \tag{27}$$

where $\lambda_j = (\Omega^2/4 + h_x^2 g_j^2)^{1/2}$. Expression (27) shows that for hard pulses ($|h_x g_j| \gg \Omega/2$) and $\lambda_j t = \pi$, the effect of the pulse is to change qubit j from $|0\rangle$ to approximately $|1\rangle$. Note however that a sequence of such pulses can never turn $|0\rangle$ into $|1\rangle$ exactly and that both spins are affected by the pulse.